

Symplectic structures on the tangent bundle of a smooth manifold

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Abstract

We give a method to lift $(2,0)$ -tensors fields on a manifold M to build symplectic forms on TM . Conversely, we show that any symplectic form Ω on TM is symplectomorphic, in a neighborhood of the zero section, to a symplectic form built naturally from three $(2,0)$ -tensor fields associated to Ω .

1 Introduction

The geometry of the tangent bundle TM of a Riemannian manifold with the Sasaki metric has been extensively studied since the 60's (see [1, 9]). To overcome the rigidity of these metrics many others metrics generalizing Sasaki metrics were introduced and studied (see [6]). Given a Riemannian manifold (M, g) , the basic idea behind the construction of Riemannian metrics on TM from the Riemannian metric on M is to use the Levi-Civita connection of g to get a splitting of $TTM = \mathcal{V}M \oplus \mathcal{H}M$ and to lift the metric g to TM by the mean of this splitting. It is natural, in order to construct symplectic structures on the tangent bundle, to use the same approach in the case where Riemannian metrics are replaced by differential 2-forms or, more generally, $(2,0)$ -tensor fields. This point of view has been adopted in [4, 5, 7] motivated by the classification of "natural" symplectic forms on the tangent bundle. In this paper, we address the two following situations:

1. Starting with a manifold M endowed with two differential 2-forms ω_0, ω_1 , a $(2,0)$ tensor field A and a linear connection ∇ , we construct a natural differential 2-form Ω on TM involving the ω_i , A and the splitting of TTM induced by ∇ . We give then the sufficient and necessary conditions on ω_i , A and ∇ for which Ω is symplectic (see Proposition 3.2). Among these conditions, (A, ∇) satisfy an equation which is known as Codazzi equation (see [3, 10]) when A is a Riemannian metric

and ∇ flat. We give in Proposition 3.1 some equivalent assertions to this equation.

2. Conversely, to any symplectic form Ω on TM , we associate two differential 2-forms $\omega_{11}, \omega_{22} \in \Omega^2(M)$ and $(2, 0)$ -tensor fields A on M . We show that, for any choice of a connection ∇ on M , Ω is symplectomorphic, near the zero section, to a symplectic form built from $(\nabla, \omega_{11}, \omega_{22}, A)$ in a way described in (i) (see Theorem 2.1).

The paper is organized as follows. In Section 2 we state our main result and in Section 3, we define the lift of $(2, 0)$ -tensor fields to the tangent bundle by the mean of a linear connection and we prove Propositions 3.1-3.2. Section 4 is devoted to a proof of Theorem 2.1 which is mainly based on a version of the classical Darboux's Theorem.

2 Statement of the main result

Let M be a manifold and $\pi : TM \rightarrow M$ its tangent bundle. We denote by $\iota : M \rightarrow TM$ the zero section and by $\mathcal{V}M = \ker d\pi$ the vertical subbundle of TM . For any $x \in M$ and $u \in T_x M$ there is a natural isomorphism $\tau_{(x,u)} : T_x M \rightarrow \mathcal{V}_u M$ given by $\tau_{(x,u)}(v) = \frac{d}{dt}|_{t=0}(u + tv)$. For any vector field X on M , we define its *vertical lift* X^v which is the vector field on TM given by $X^v(x, u) = \tau_{(x,u)}(X_x)$. On the other hand, for any $x \in M$, we have $T_{0_x} TM = \mathcal{V}_{0_x} M \oplus \iota_*(T_x M)$, where 0_x is the null vector of $T_x M$. For any differential 2-form Ω on TM , we associate three $(2, 0)$ -tensor fields $\omega_{11}, \omega_{22}, A$ on M by putting

$$\begin{aligned} \omega_{11} &= \iota^* \Omega, \quad A(u, v)(x) = \Omega(\tau_{(x,0_x)}(u), \iota_*(v))_{0_x}, \\ \omega_{22}(u, v)(x) &= \Omega(\tau_{(x,0_x)}(u), \tau_{(x,0_x)}(v))_{0_x}. \end{aligned} \quad (1)$$

Suppose now that M carries a linear connection ∇ . This define an horizontal distribution on TM as follows:

$$\mathcal{H}_{(x,u)} M = \left\{ \frac{d}{dt}|_{t=0} P_{c_x}^u(t), c_x \in C^\infty(]-\epsilon, \epsilon[, M) \text{ and } c_x(0) = x \right\}$$

where $P_x^u(t) :]-\epsilon, \epsilon[\rightarrow TM$ is the parallel transport with respect to ∇ of u along the curve c_x . The linear map $T_{(x,u)}\pi : \mathcal{H}_{(x,u)} M \rightarrow T_x M$ is an isomorphism and hence $TM = \mathcal{V}M \oplus \mathcal{H}M$. For any vector field X on M , we define its horizontal lift X^h by $X_{(x,u)}^h = (T_{(x,u)}\pi)^{-1}(X_x)$. For any $\omega \in \Omega^2(M)$

we define the differential 1-form $\lambda^{\omega, \nabla}$ on TM by putting, for any vector field X on M and for any $u \in TM$,

$$\lambda^{\omega, \nabla}(X^h) = 0 \quad \text{and} \quad \lambda^{\omega, \nabla}(X^v)(u) = \frac{1}{2}\omega(u, X). \quad (2)$$

We can now state our main result.

Theorem 2.1 *Let Ω be a symplectic form on TM . Let $\omega_{11}, A, \omega_{22}$ be the associated $(2, 0)$ -tensor fields given by (1). Then, for any linear connection ∇ , there exists two open neighborhoods \mathcal{N}_1 and \mathcal{N}_2 of the zero section in TM and a diffeomorphism $\phi : \mathcal{N}_1 \longrightarrow \mathcal{N}_2$ such that*

$$\phi|_{\iota(M)} = \text{Id}_{\iota(M)} \quad \text{and} \quad \phi^*\Omega = \pi^*\omega_{11} + (A^\flat)^*(d\lambda) + d\lambda^{\omega_{22}, \nabla},$$

where λ is the Liouville 1-form on T^*M and $A^\flat : TM \longrightarrow T^*M$ is given by $A^\flat(u) = A(u, \cdot)$.

3 Lift of $(2, 0)$ -tensor fields on M to symplectic forms on TM

Let M be a manifold endowed with a linear connection ∇ and $TTM = \mathcal{V}M \oplus \mathcal{H}M$ the associated splitting. Let (x^1, \dots, x^n) be a local coordinates system on M and $(x^1, \dots, x^n, u^1, \dots, u^n)$ the corresponding coordinates system on TM . Let (Γ_{ij}^k) the Christoffel's symbols of ∇ defined by $\nabla_{\partial_{x^i}} \partial_{x^j} = \sum_{k=1}^n \Gamma_{ij}^k \partial_{x^k}$. If $X = \sum_{i=1}^n X^i \partial_{x^i}$ then

$$X^h = \sum_{i=1}^n X^i \partial_{x^i} - \sum_{i,j,k} \Gamma_{ij}^k u^i X^j \partial_{u^k} \quad \text{and} \quad X^v = \sum_{i=1}^n X^i \partial_{u^i}. \quad (3)$$

We deduce easily from these formulas:

$$\begin{aligned} [X^h, Y^h] &= [X, Y]^h - (R(X, Y)u)^v, & [X^v, Y^v] &= 0, \\ [X^h, Y^v] &= (\nabla_X Y)^v, \end{aligned} \quad (4)$$

where R is the curvature of ∇ given by $R(X, Y) = \nabla_{[X, Y]} - (\nabla_X \nabla_Y - \nabla_Y \nabla_X)$. Let ω_0, ω_1 two differential 2-forms on M and A a $(2, 0)$ -tensor field. We define $\Omega \in \Omega^2(TM)$ by

$$\begin{aligned} \Omega(X^v, Y^v) &= \omega_0(X, Y) \circ \pi, & \Omega(X^h, Y^h) &= \omega_1(X, Y) \circ \pi, & \Omega(X^v, Y^h) &= A(X, Y) \circ \pi, \\ \Omega(X^h, Y^v) &= -A(Y, X) \circ \pi. \end{aligned} \quad (5)$$

We call Ω the *lift* of $(\nabla, \omega_0, \omega_1, A)$. This notions appeared in [4] when $\omega_0 = \omega_1 = 0$ and A is a Riemannian metric. It is obvious that Ω is nondegenerate iff, for any local coordinates system (x^1, \dots, x^n) the matrix $\begin{pmatrix} P & M \\ -M^t & Q \end{pmatrix}$, where $P = (\omega_{11}(\partial_{x^i}, \partial_{x^j}))_{1 \leq i, j \leq n}$, $Q = (\omega_{22}(\partial_{x^i}, \partial_{x^j}))_{1 \leq i, j \leq n}$, $M = (A(\partial_{x^i}, \partial_{x^j}))_{1 \leq i, j \leq n}$ is invertible. A direct computation using (4) gives:

$$\begin{aligned} d\Omega(X^h, Y^h, Z^h) &= d\omega_1(X, Y, Z) \circ \pi + \oint A(R(X, Y)u, Z) \circ \pi, \\ d\Omega(X^v, Y^v, Z^v) &= 0, \\ d\Omega(X^v, Y^v, Z^h) &= \nabla_Z \omega_0(X, Y) \circ \pi, \\ d\Omega(X^h, Y^h, Z^v) &= -\nabla_X A(Z, Y) \circ \pi + \nabla_Y A(Z, X) \circ \pi + A(Z, \tau(X, Y)) \\ &\quad + \omega_0(R(X, Y)u, Z) \circ \pi, \end{aligned} \tag{6}$$

where τ is the torsion of ∇ given by $\tau(X, Y) = [X, Y] - \nabla_X Y + \nabla_Y X$. We call the equation

$$\nabla_X A(Z, Y) - \nabla_Y A(Z, X) = A(Z, \tau(X, Y)) \tag{7}$$

Codazzi equation. Indeed, when ∇ is torsion free and A is a pseudo-Riemannian metric, we recover the Codazzi equation known in the context of Hessian manifolds ([10]). It appeared also in [3]. The following result is a generalization both of a result of Delanoë [3] and a result by Janyska in [4, 5].

Proposition 3.1 *Let (M, ∇) be a manifold endowed with a connection and A a nondegenerate $(2, 0)$ -tensor field. Let Ω be the lift of $(\nabla, 0, 0, A)$. Then the following assertions are equivalent:*

1. (A, ∇) satisfies Codazzi equation (7).
2. $\Omega = (A^\flat)^*(d\lambda)$, where λ is the Liouville 1-form on T^*M and $A^\flat : TM \rightarrow T^*M$ is given by $A^\flat(u) = A(u, \cdot)$.
3. Ω is symplectic.
4. $A^\flat(\mathcal{H}M)$ is Lagrangian with respect to $d\lambda$, where λ is the Liouville 1-form on T^*M and $\mathcal{H}M$ is the horizontal distribution associated to ∇ .

Proof. Remark first that since A is nondegenerate then Ω is nondegenerate. We choose a local coordinates system $(x^i)_{i=1}^n$ and we denote by $(x^i, u^i)_{i=1}^n$ and $(x^i, p^i)_{i=1}^n$ the corresponding coordinates on TM and T^*M respectively. The Liouville 1-form is given by $\lambda = \sum_{i=1}^n p^i dx^i$ and A^\flat is given by

$A^b(x^1, \dots, x^n, u^1, \dots, u^n) = (x^1, \dots, x^n, P^1, \dots, P^n)$, where $P^i = \sum_{j=1}^n u^j A_{ji}$, with $A_{ij} = A(\partial_{x^i}, \partial_{x^j})$. Thus $(A^b)^*(d\lambda) = \sum_{i=1}^n dP^i \wedge dx^i$. By using (3), we get for $i = 1, \dots, n$,

$$\partial_{x^l}^h = \partial_{x^l} - \sum_{j,k} \Gamma_{jl}^k u^j \partial_{u^k}, \quad \text{and} \quad \partial_{x^l}^v = \partial_{u^l}.$$

Thus $\partial_{x^l}^h(P^s) = \sum_{j=1}^n u^j (\partial_{x^l}(A_{js}) - \sum_k \Gamma_{jl}^k A_{ks})$. Now

$$\begin{aligned} (A^b)^*(d\lambda) (\partial_{x^l}^v, \partial_{x^s}^v) &= 0, \\ (A^b)^*(d\lambda) (\partial_{x^l}^h, \partial_{x^s}^h) &= \sum_{i=1}^n (\partial_{x^l}^h(P^i) \partial_{x^s}^h(x^i) - \partial_{x^l}^h(x^i) \partial_{x^s}^h(P^i)) \\ &= \partial_{x^l}^h(P^s) - \partial_{x^s}^h(P^l) \\ &= \sum_{j=1}^n u^j \left(\partial_{x^l}(A_{js}) - \partial_{x^s}(A_{jl}) - \sum_k (\Gamma_{jl}^k A_{ks} - \Gamma_{js}^k A_{kl}) \right) \\ (A^b)^*(d\lambda) (\partial_{x^l}^v, \partial_{x^s}^h) &= \sum_{i=1}^n (\partial_{x^l}^v(P^i) \partial_{x^s}^h(x^i) - \partial_{x^l}^h(x^i) \partial_{x^s}^h(P^i)) = \partial_{x^l}^v(P^s) = A_{ls}. \end{aligned}$$

This shows that (i), (ii) and (iv) are equivalent. Moreover, (ii) implies (iii) obviously and the expression $d\Omega(X^h, Y^h, Z^v)$ given in (6) shows that (iii) implies (i). \square

Proposition 3.2 *The differential 2-form Ω is closed if and only if the following relations hold:*

1. $d\omega_1 = 0$, $\nabla\omega_0 = 0$ and, for any X, Y, Z, T , $\omega_0(R(X, Y)Z, T) = 0$,
2. (∇, A) satisfy the Codazzi equation (7).

Proof. If Ω is closed then, according to the relations (6), (i) and (ii) hold. Conversely, write $\Omega = \Omega_1 + \Omega_2$ where Ω_1 is the lift of $(\nabla, \omega_0, \omega_1, 0)$ and Ω_2 is the lift of $(\nabla, 0, 0, A)$. So if (i) and (ii) hold then, by Proposition 3.1, $\Omega_2 = (A^b)^*(d\lambda)$ which is closed. Hence Ω is closed iff Ω_1 is closed which is guaranteed by (i). \square

Let us give some situations where we can use Proposition 3.2 or Proposition 3.1 to build symplectic forms on the tangent bundle or Lagrangian horizontal distribution on T^*M .

Example 1 1. Let (M, ω_1) be a symplectic manifold, ω_0 a nondegenerate 2-form on M and ∇ a flat connection such that $\nabla\omega_0 = 0$. According to Proposition 3.2 the lift of $(\nabla, \omega_0, \omega_1, 0)$ is a symplectic form on TM .

2. Let G be a Lie group, ω_1 a left invariant symplectic form on G , ω_0 a nondegenerate right invariant 2-form on G , ∇ the flat connection on G satisfying $\nabla X = 0$ for any right invariant vector field. Then, according to Proposition 3.2, the lift of $(\nabla, \omega_0, \omega_1, 0)$ is a symplectic form on TG .
3. Let M be a manifold, ∇ a connection on M and α a differential 1-form on M . Put $A(X, Y) = \nabla_Y \alpha(X)$. One can check easily that

$$\nabla_X A(Z, Y) - \nabla_Y A(Z, X) = A(Z, \tau(X, Y)) + \alpha(R(X, Y)Z).$$

So if ∇ is flat then (A, ∇) satisfy Codazzi equation. By choosing α such that for any coordinates system (x^1, \dots, x^n) on M the matrix $(\nabla_{\partial_{x^i}} \alpha(\partial_{x^j}))_{1 \leq i, j \leq n}$ is invertible and by using Proposition 3.1, we get that $A^b(\mathcal{H})$ is a Lagrangian distribution with respect to $d\lambda$.

We give now an example of ∇ and α satisfying the conditions above. We consider \mathbb{R}^n with its canonical ∇ and let $B = (b_{ij})$ be an invertible n -matrix. Put, for $i = 1, \dots, n$, $\alpha(\partial_{x^i}) = \exp(\sum_{k=1}^n b_{ki} x^k)$. An easy computation gives that

$$(\nabla_{\partial_{x^i}} \alpha(\partial_{x^j}))_{1 \leq i, j \leq n} = BD$$

where D is the diagonal matrix with entries $\alpha(\partial_{x^i})$, $i = 1, \dots, n$.

4. Let (M, g) be a pseudo-Riemannian manifold and ∇ the Levi-Civita connection of g . According to Proposition 3.1, the lift of $(\nabla, 0, 0, g)$ is $(g^b)^*(d\lambda)$. This situation was pointed out in [4, 5]. Moreover, $g^b(\mathcal{H}M)$ is an horizontal Lagrangian distribution.
5. Let (M, ω) be a symplectic manifold and ∇ a torsion free connection such that $\nabla \omega = 0$. It is a well-known result that there are many such connections (see [2]). According to Proposition 3.1, the lift of $(\nabla, 0, 0, \omega)$ is $(\omega^b)^*(d\lambda)$ and $\omega^b(\mathcal{H}M)$ is an horizontal Lagrangian distribution.

4 Proof of Theorem 2.1

The proof of Theorem 2.1 is based on the following version of the classical Darboux's theorem (see [8]).

Theorem 4.1 *Let V be a smooth manifold and $\omega_1, \omega_2 \in \Omega^2(M)$ are closed. Suppose that N is a submanifold of V such that for any $q \in N$, $\omega_1(q) = \omega_2(q)$*

and $\omega_1(q), \omega_2(q)$ are non-degenerate. Then there exists two open neighborhoods \mathcal{N}_1 and \mathcal{N}_2 of N and a diffeomorphism $\phi : \mathcal{N}_1 \rightarrow \mathcal{N}_2$ such that

$$\phi|_N = \text{Id}_N \quad \text{and} \quad \phi^* \omega_2 = \omega_1.$$

Proof of Theorem 2.1. Put $V = TM$, $N = \imath(M)$, $\omega_1 = \Omega$ and $\omega_2 = \pi^* \omega_{11} + (A^b)^*(d\lambda) + d\lambda^{\omega_{22}, \nabla}$ and apply Darboux's theorem. The key point is to check that ω_1 and ω_2 agree on the zero section. This is a consequence of the expression of $(A^b)^*(d\lambda)$ computed in the proof of Proposition 3.1 and the following formulas:

$$\begin{aligned} d\lambda^{\omega_{22}, \nabla}(X^h, Y^h)(u) &= \frac{1}{2} \omega_{22}(R(X, Y)u, u), \\ d\lambda^{\omega_{22}, \nabla}(X^v, Y^v) &= \omega_{22}(X, Y) \circ \pi, \\ d\lambda^{\omega_{22}, \nabla}(X^h, Y^v)(u) &= \frac{1}{2} \nabla_X \omega_{22}(u, Y). \end{aligned}$$

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